1. EUCLID’S AXIOMS

§1.1. Euclid’s Academy

Mathematics can’t claim to be the world’s oldest profession. But as an intellectual activity it is certainly one of the oldest. Of course mathematics only became a profession around the time of the Renaissance. But historians believe that mathematics has been practised for many thousands of years.

The motivation was practical. It was to serve the needs of commerce. There were only two branches of mathematical knowledge back then: arithmetic and geometry.

Arithmetic was developed in order to support book-keeping (though records of financial transactions were written on stone, or papyrus sheets, not books).

Geometry was developed as an aid to surveying. The word 'geometry' comes from the Greek, meaning 'measuring the earth'.

Euclid, in the 3rd or 4th century BC, was a Greek who is credited with making a systematic intellectual discipline out of the many rules of thumb that were previously in use. This was at a time long before universities and it is believed by historians that Euclid surrounded himself by disciples, probably much younger than himself. He ran something between an academy and a research school.

One imagines them sitting around a sandy square in Athens, drawing diagrams in the sand and debating geometric ideas. They may have used the Socratic method where dialogue and discussion was used to locate truths.

It has been said that, before Euclid, geometry employed the scientific method. Perhaps Pythagoras formulated his famous theorem by examining a large number of right-angled triangles. That's how we might do it today, but papyrus was scarce back then. It was long known that the 3-4-5 triangle was right-angled and somebody might have stumbled on the 5-12-13 example. Then perhaps somebody else noted the arithmetic pattern in these numbers, but a proof was still to come.

No doubt the discussion that took place between the Euclideans involved a lot of argument along the lines of 'surely …' or 'it would seem reasonable that …'. But I imagine that short arguments would have been put forward that provided logical bridges between some of these geometrical statements. “Well we all know that … and so it follows that … (perhaps with a few extra construction lines) …”.

We have no way of knowing what went on in these discussions, but I can imagine Euclid himself coming up with the idea of systematising all these bridges and creating a unified structure that built geometry from a small number of postulates, or axioms. These were very basic statements which could be accepted intuitively. For example, “given any two distinct points there exists exactly one straight line passing through them”. Perhaps this would have been backed up by a small amount of experimentation, but I’m sure you’ve seen enough examples to know in your heart that it’s true. Of course you probably never considered the possibility that there might be many ‘straight’ lines joining them that were so close to each other that your eye couldn’t tell the difference.

Euclid’s *magnum opus* is his *Elements*. This has been a standard text-book in universities and schools throughout many centuries. It was used, both in the original Greek, and later in translation, up until the end of the 19th century. It is said that it is second only to the Bible in the number of editions (over a thousand) printed since the first printed edition in 1482.

§1.2. The Role of Intuition

Euclid had the vision of formulating geometry in such a way that the truth of the theorems didn't rest on the intuition of the individual. By setting down axioms, and building everything logically from these axioms, everyone who accepted the axioms would have to accept all the theorems. And these axioms were considered to be self-evident.

The problem was that he used this ground-breaking approach to mathematics on geometry, where intuition normally plays such an important part. Angles are the most difficult part to axiomatize and he relied on the diagrams in certain places.

Consider the difficulty that angles create. It's clear, from the fact that they are sometimes added, that they have to be numbers of some sort. But if you have three lines, and hence three pairs of lines and hence three angles, it's not clear, without a diagram, which two are added to give the third. The angles need to have a sign (positive or negative) and this requires the notion of clock-wise and anti-clockwise rotations.

I believe that a proper account of any branch of mathematics should avoid any need for mathematical, and in particular geometric, intuition. That's not to say that intuition plays no role in mathematics, far from it. A good mathematical exposition should make use of the reader's intuition to help him or her understand the formal proof. But, at least with advanced students, the formal proof should be capable of standing alone.

Mathematical intuition is in fact what drives mathematical research. No mathematician ever discovers his theorems by playing with axioms. He or she finds them by use of a highly developed intuition. But as valuable as that intuition is, mathematicians don't stop until they can translate their proof into a formal series of deductions that can stand without their intuition.

There was once an Indian called Ramanujan who amazed some Oxford professors with his amazing intuition in the area of infinite series. You may have seen his story in the movie *The Man Who Knew Infinity*.

He knew intuitively a huge number of new mathematical results but he had a poorly developed concept of proof. He claimed it was an Indian goddess who revealed these mathematical truths to him. But whenever he announced some new result, the professors went away and always came up with a proof.

So, my criterion for a thoroughly rigorous development of geometry, or any other branch of mathematics for that matter, is that it should be intelligible to a disembodied angel.

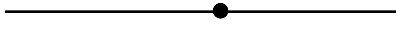
§1.3. The Disembodied Angel

Years ago one of my colleagues at Macquarie University, Alan Macintosh, invented a pedagogical tool called *The Disembodied Angel*. He died a long time ago and is probably now a disembodied angel himself.

The disembodied angel was an imaginary creature who was highly intelligent but who had no spatial sense.

It lived in a spiritual realm and had no concept of geometrical entities.

Alan had a pair of walkie talkies (these days we'd use mobile phones). One student went into another room with one of the walkie talkies and played the part of the disembodied angel. He had to pretend he had no geometrical intuition. Another student went to the board, in the lecture room, and tried to describe the following geometric construction to the angel in the other room.



Student: Well you've got two points which lie on a line.

Angel: I understand everything, except ‘point’, ‘line’ and ‘lie on’.

Student: Well a point is like a dot.

Angel: I’ve never encountered a dot. Is it like a cherubim?

Student: No, it’s something that has no length or breadth.

Angel: I don’t know what ‘length’ and ‘breadth’ mean. What about a line?

Student: It’s something that’s infinitely long but has zero breadth.

Angel: Oh, I understand ‘infinite’. God is infinite. And I’m good with numbers, three for Trinity, you know. Zero? Yes I remember a newcomer to Heaven once asking the Archangel Gabriel how many sins I had committed and Gabriel said, “zero”. I think ‘zero’ is another way of saying “none”.

Student: Yes that’s right. We’re getting somewhere at last.

Angel: But I still don’t know about ‘length’ and ‘breadth’.

Student: Well never mind. Just accept them as undefined entities.

Angel: Fine, but what about ‘lie on’. It sounds like some sort of relation.

Student: Yes, there’s an undefined relation of a point lying on a line.

Angel: I’m fine with that too. Can a point lie on more than one line?

Student: Oh yes, all the time. Now you take a third point that doesn’t lie on this line.

Angel: Got it.

Student: Well Euclid says there’s exactly one line that passes through that third point and is parallel to the first line.

Angel: I presume Euclid is your friend. And I’m OK with ‘exactly one’. I know that there is exactly one God. But I’m puzzled by ‘passing through’.

Student: Oh that’s easy. To say that a line passes through a point is just another way of saying that the point lies on the line.

Angel: Great. So all I need now is to understand ‘parallel’. Is it another undefined relation, this time a relation between two lines?

Student: No. Two lines are parallel if they remain a constant distance from one another.

Angel: Distance?

Student: How about if I say that parallel lines are those where the angle between them is zero?

Angel: Hmm. The angel between two lions is zero. God has been described as the Lion of Judah but I can't think who the other one might be.

Student (starting to become frustrated): No! Let me put it another way. The two lines don't meet.

Angel: Meet? I've heard of that in the mass. "It is right and meet so to do."

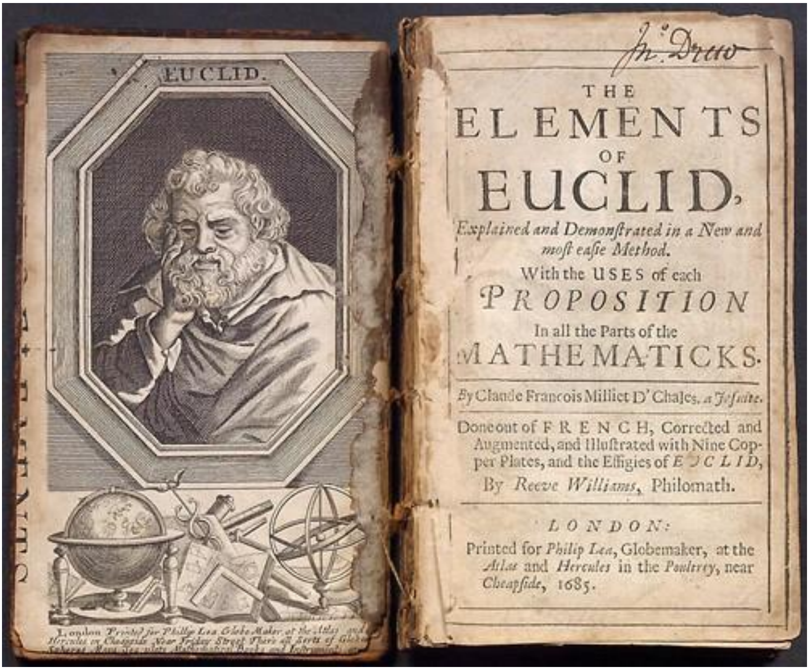
Student: It's nothing like that. How can I put it? Two lines are parallel if there is no point that lies on both lines.

Angel: Oh now I get it. And that statement you said at the beginning. Is it true?

Student: Euclid says so. It's one of his postulates.

The point of this story is to explain that the modern abstract approach to mathematics is to set up an axiomatic creed – undefined entities, definitions, and axioms. The subject development follows by using logic to prove theorems. It makes no use of intuition or diagrams and should be intelligible to a disembodied angel.

§1.4. Euclid’s Axiomatic Account



Euclid began with a collection of definitions. In an axiomatic system there are two types of **definition**. There are the undefined entities and then there are true definitions that are built up from these.

Points and lines, and the relation of a point lying on a line (or the line passing through a point) should be considered as undefined entities. However Euclid defines a point as “that which has no part” and a line as “a length without breadth". Both of these so-called definitions would be useless to a disembodied angel. Far better to have them as undefined entities. Euclid doesn't appear to define the relation of 'lying on' but that can be taken to be another undefined entity.

To Euclid, a line is always finite and has two endpoints, and the line with endpoints A and B is denoted by AB. But in postulate (2) he asserts that "a straight line can be produced continuously in a straight line". This means that two lines (except for parallel ones) will intersect in a point, even if that point lies beyond the endpoints. Far simpler is to consider all lines as infinite.

He defines a circle to be "plane figure contained by a single line, called the **circumference**, such that all of the straight lines radiating towards the circumference from one point (called the **centre**) amongst those lying inside the figure are equal to one another".

Clearly he means 'equal in length', rather than identical. So we have to capture the notion of equality of line intervals, or equivalently the distance between two points.

None of this would make sense to a disembodied angel. The whole point of having axioms is that we don't have to have any preconceived knowledge of these things. You and I might think of points and lines as marks on paper. But for all we know the disembodied angel might think of points as cherubim and lines as sins they have committed.

So P lies on *h* might mean that cherubim P has committed the transgression *h*. And parallel lines might be different sins where no cherubim has committed both. That won't matter as long as the angel doesn't rely on that interpretation.

In fact once the axioms have been stated she may realise that that interpretation won't work and she'll just shrug her wings and say, "I haven't the faintest idea what these things are, but I can still prove theorems about them!"

Now Euclid had the right idea but the disembodied angel would have some difficulties. Euclid is very patchy when it comes to lengths and angles. It would appear that these are like numbers, in so far as he says things like

$$AB < CD \text{ or } \angle ABC > \angle DEF.$$

So it would appear that there is an ordering on line segments and angles. We would have to bring the angel up to speed on partial orderings.

Euclid is pretty vague about this, apart from saying that "the whole is greater than the part".

When it comes to angles the level of difficulty escalates. He defines 'angle' as the inclination of one line with another'. That seems to be defining one word in terms of a synonym. It could be taken as yet another undefined concept.

The biggest difficulty with angles is to define them in such a way that we can add them. When three lines meet at a point we have three angles and we have to prove that two angles add to the third. But which two? If we had such a situation without a diagram how would we know which two angles add up to the third?

Euclid states five ‘postulates’. We would call them ‘axioms’. They are:

(E1) Through any two points there passes exactly one line. He doesn’t say ‘exactly one’ but he appears to assume it.

(E2) Any finite line can be produced continuously in a straight line.

Euclid thinks of his lines as having endpoints and this axiom says that these lines can be extended as needed. It is much simpler to imagine that all lines are infinite. Such lines don’t come with any named points on them.

(E3) Given a point, and a radius, there exists a circle with that point as centre and with that radius.

(E4) All right angles are equal to one another.

It’s not quite clear what he means exactly. Equal in what sense?

But now we come to the most important axiom of all – one that sets Euclidean geometry from non-Euclidean geometries. It is the Parallel Postulate. Euclid states it as:

(E5) If a straight line, falling across two other straight lines, makes internal angles on the same side whose sum is less than two right angles, then the two other straight-lines, being produced to infinity, meet in a point.

An alternative to this formulation was given by John Playfair (1748-1819).
(P5) Given a line *h* and a point P not on *h* there exists exactly one line through P that is parallel to *h*.
The advantage of this version is that avoids the tricky bit about angles. I'll prove in a later chapter that Playfair's version is equivalent to Euclid's.

Now I don't want to give the impression that Euclid did a pretty poor job. He was, in fact, far ahead of his time. But centuries of playing with axiomatic systems enable us to improve on his remarkable formulation of geometry.

My approach, rather than patch up Euclid, is to formulate Euclidean Geometry, along with the rest of mathematics, entirely within set theory. That is, I assume nothing further than the axioms for set theory. I have given a full account of this in my notes on *Set Theory*.

I first construct the real numbers, as sets, and develop all the usual arithmetic of real numbers. I then define a **point** as a complex numbers and a **line** as a set of points. P **lies on** a line *h* simply means that P $\in$ *h*. Two lines are parallel if there is no point that lies on both. We write $h \parallel k$ if h is parallel to k, although we could equally well write $h \cap k = \emptyset$.

I will adopt the convention that a point will be written with a capital letter. If I wish to consider it as a vector I will use the corresponding lower case letter, written bold. If I wish to consider it as a complex number I will write it as a lower case letter, unbolded but in italics.

So P represents a point. It will be written as **p** if I wish to consider it as a vector and p if I want to consider it as a complex number.

The **distance** between two points P, Q is $|p - q|$ (the modulus of the difference). We can also write it as $|\mathbf{p} - \mathbf{q}|$ (length of a vector).

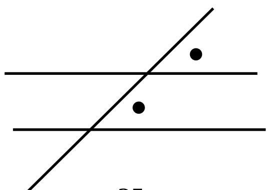
We define a circle as a set of points, P, that satisfy an equation of the form $|\mathbf{p} - \mathbf{c}| = r$ for some fixed point C, called the **centre**, and some positive r, called the **radius**.

Angles are defined by using arguments of complex numbers. If A, B, C are distinct points (complex numbers) then the **angle**

$$\angle \mathbf{ABC} = \arg(c - b) - \arg(a - b).$$

This quantity is to be considered as an element of the real numbers modulo 2π.

It follows from this definition that if a line cuts a pair of parallel lines then corresponding angles are equal.



It was appropriate to use radian measure while we were using the trigonometric functions, but in the following we'll use 180° instead of π. Of course Euclid didn't know about degrees. He took right angles as his unit of measurement, which we identify as 90°.

We write AB ⊥ CB if ∠ABC = 90°. Since arg i = π/2, multiplication by i rotates a complex number anti-clockwise through and angle of 90°. It is easy to see that this corresponds to orthogonal vectors. So if AB ⊥ CB then $(\mathbf{a} - \mathbf{b}).(\mathbf{c} - \mathbf{b}) = 0$.

But often we prove geometric theorems algebraically. Sometimes we view a point as a pair of real numbers, sometimes as a single complex number, and sometimes as a vector. For example, if U, V are points represented by the vectors **u**, **v**, the distance UV is defined as $\sqrt{(\mathbf{u} - \mathbf{v}).(\mathbf{u} - \mathbf{v})}$ we can present a purely algebraic proof.

Angles are difficult enough to explain to a disembodied angel, but consider the concept of area. Now you may be thinking, "who cares about the disembodied angel?" But remember, if we want to put mathematics onto a firmly logical foundation (well, as firmly logical as a collection of set theory axioms can be if we can't even prove them to be consistent) then we have to address these problems.

What exactly is area? We seem to have an intuitive concept of area, possibly based on the notion that, for paper shapes, area and weight are related and we could determine the area of an irregular shape by weighing it. Of course we will have left our disembodied angel far behind!

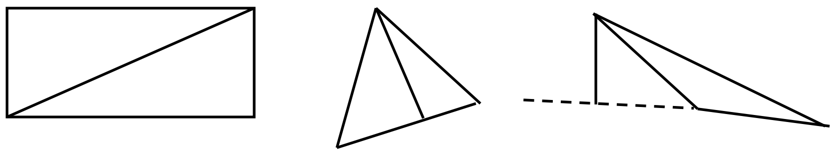
We have the belief that areas are additive – that the area of the sum of the disjoint union of two shapes is the sum of their areas. But this seems to be an assumption – that is, an extra axiom.

What in fact is a shape anyway? It is more than just a subset of the Euclidean plane. Some subsets would have to be considered to have infinite area. But there are finite shapes where the concept of area is problematical, for example the set of all points in the unit circle that have rational coordinates. Try cutting that shape out of a piece of paper!

There is an area of mathematics that tackles this problem head on – Measure Theory. We might have to teach our angel some measure theory! But, as yet, I have no notes on this branch of mathematics so you'd have to look beyond coopersnotes.net.

In primary school we were taught that the area of a rectangle is 'base times height'. The fact that areas are additive was just assumed. So the area of a right-angled triangle, being half a rectangle, would be 'half the base times the perpendicular height', provided the base wasn't the hypotenuse.

Then, since any triangle can be broken up into two right-angled triangles, we saw that the area of any triangle is 'half the base times the perpendicular height'.



Note that the proof is slightly different in the two cases of a triangle with only acute angles and one with an obtuse angle.

We didn't address the possibility that this might depend on which side of the triangle is taken as the base. And we use the unstated axiom that area of a polygon is the sum of the triangular pieces when the polygon is triangulated.

Do I hear you say, "why not take this as the *definition* of area, at least for polygons? That would avoid having to assume that area is additive". The problem is that we'd have to *prove* that such a definition is independent of the way the shape is triangulated.

The point of all this is to show that, like arithmetic, geometry is very difficult to discuss rigorously. But you may be eager to put all these difficulties behind you and start learning some actual geometry. A small amount of what you will find in subsequent chapters you learnt at school. The majority of it may well be new to you.

So, we leave the disembodied angel behind and, following in Euclid's footsteps, we make great use of a variety of techniques. I have tried to convince you that Euclidean Geometry *could* be taught to a disembodied angel – but we don't have to actually do it!

So in the following chapters I shall make use of whatever approach gives the simplest, or most elegant proof of a particular theorem. Here is our tool-bag.

(1) Geometric arguments along the lines of Euclid
Often the most elegant solution is based on a diagram, though we have to be careful that we include all cases, such as triangles having an obtuse angle and triangles with only acute angles.

(2) Cartesian geometry
Introducing coordinates, and using basic algebra, can often lead to the simplest proof. For example Euclid spends half a page showing that two concentric circles don't intersect! This is so obvious using coordinates that it hardly warrants a proof. But beware. Cartesian geometry is not a panacea. In many cases it just leads to intractable algebra.

(3) Complex numbers
A point in the Euclidean plane can be represented by a complex number and in a few situations that can be useful.

(4) Vector algebra

Representing points by vectors, and using the dot product for lengths and perpendicularity, can sometimes give dramatically simpler solutions than other techniques.

(5) Calculus

Calculus is very useful when dealing with tangents.

(6) Trigonometry

There is no real need to make use of trigonometry, but it is convenient to use the trigonometric functions in cases such as similar triangles.

(7) Linear algebra

Linear transformations preserve certain geometric properties such as parallelism and midpoints. Sometimes a theorem can be proved in a more general setting by applying a linear transformation to change that general situation into a simpler one. For example, ellipses can be transformed into circles by a simple linear transformation and the general hyperbola can be transformed into the rectangular hyperbola $y = \frac{1}{x}$, leading to simpler proofs.

(8) Projective Methods

Theorems that only involve incidence properties (points lying on lines) are most elegantly proved using the techniques of projective geometry. Here I avoid proving such theorems because it takes a fair bit of time to become proficient in these methods and I don’t want to encumber you with the clumsy proofs that would be otherwise necessary. You can find a full discussion of Projective Geometry (from a linear algebra perspective) in my notes *Geometry vol 2*.